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ON THE MOTION OF A SOLID IN A MAGNETIC FIELD*

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The problem of the motion of a magnetic solid in a constant uniform magnetic field, taking gyromagnetic effects into account, is considered. The equations of motion are derived, the Hamiltonian structure is studied, and the cases of integrability indicated. Certain classes of stationary motions are studied and their stability examined.

The gyromagnetic effects arise because the electrons have magnetic and mechanical spin moments /1/. The rotation of the body causes it to become magnetized (the Barnett effect) and when a freely suspended body is magnetized, it begins to rotate (the Einsteinde Haas effect). It is found that gyromagnetic phenomena must be taken into account when analysing the motion of gyroscopic precision systems.

1. The construction of models of continua possessing gyromagnetic properties involves introducing into the defining parameters the internal angular momentum /3, 4/. Let a solid move through an unbounded volume of ideal incompressible fluid which is at rest at infinity, and in a constant uniform magnetic field $\mathbf{h}$. We will assume that the velocity of the fluid allows of a single-valued potential. The free-energy density within the volume G occupied by the body has the form /4/

$$F = \mathbf{B}^2/(8\pi) - \mathbf{M}\cdot\mathbf{B} + F_0(\mathbf{M}) + \boldsymbol{\Omega}\cdot\mathbf{M}/g \tag{1.1}$$

where $\mathbf{B}$ is the magnetic induction, $(\operatorname{div}\mathbf{B} = 0)$, $\mathbf{M}$ is the magnetic moment per unit volume, g is the gyromagnetic ratio, and $\boldsymbol{\Omega}$ is the angular velocity of rotation of the body. Then $\mathbf{H} = 4\pi\partial F/\partial\mathbf{B} = \mathbf{B} - 4\pi\mathbf{M}$ will be the magnetic field strength, $(\operatorname{rot}\mathbf{H} = 0)$, $\mathbf{k} = -\partial F/\partial\boldsymbol{\Omega} = -\mathbf{M}/g$ the volume density of the internal mechanical moment. We shall assume that the system is in thermal equilibrium, and that the energy dissipation can be neglected. Then $\partial F/\partial\mathbf{M} = 0$ and hence $\mathbf{B} = \partial F_0/\partial\mathbf{M} + \boldsymbol{\Omega}/g$.

Let the fluid be linearly magnetizable. Then the magnetic field will be given, in the magnetostatic approximation with the position and angular velocity $\boldsymbol{\Omega}$ of the body both fixed, by the following equations and boundary conditions:

$$\begin{aligned}&\operatorname{div}\mathbf{B}=0,\quad \operatorname{rot}\mathbf{H}=0;\quad [\mathbf{B}_n]=0,\ [\mathbf{H}_\tau]=0;\ \mathbf{H}-\mathbf{h}\to 0,\ |\mathbf{x}|\to\infty\\ &\mathbf{B}=\mu_2\mathbf{H},\quad \mathbf{x}\in G^-=\mathbf{R}^3\setminus G\\ &\mathbf{B}=\partial F_0/\partial\mathbf{M}+\boldsymbol{\Omega}/g,\quad \mathbf{H}=\mathbf{B}-4\pi\mathbf{M}\quad (\mathbf{x}\in G)\end{aligned} \tag{1.2}$$

where $\mathbf{h}$ is the constant uniform magnetic field strength and μ_2 is the magnetic permeability of the fluid.

Using the solutions of problem (1.2), we shall consider the free energy of the system

$$\Phi = \int\limits_{\mathbf{R}^3}\left(F - \frac{1}{8\pi}\mu_2\mathbf{h}^2\right)d\tau \tag{1.3}$$

where F is obtained from (1.1) when $\mathbf{x}\in G$, and $F = (8\pi)^{-1}\mu_2\mathbf{H}^2$ when $\mathbf{x}\in G^-$.

Let us calculate the variation of Φ when $\mathbf{h}$ and $\boldsymbol{\Omega}$ vary. According to (1.2) and (1.3),

$$\delta\Phi = \int\limits_{\mathbf{R}^3}\frac{1}{4\pi}(\mathbf{H}\cdot\delta\mathbf{B} - \mu_2\mathbf{h}\cdot\delta\mathbf{h})\,d\tau - \int\limits_G \mathbf{k}\cdot\delta\boldsymbol{\Omega}\,d\tau$$

We shall write the expression within the brackts in the first integrand in the form

$$\begin{aligned}&-(\mathbf{B}-\mu_2\mathbf{H})\cdot\delta\mathbf{h} + \mathbf{H}\cdot\delta(\mathbf{B}-\mu_2\mathbf{h}) + (\mathbf{B}-\mu_2\mathbf{h})\cdot\delta\mathbf{h} =\\ &-(\mathbf{B}-\mu_2\mathbf{H})\cdot\delta\mathbf{h} + \operatorname{div}[\delta(\mathbf{A}-\mathbf{A}_0)\times\mathbf{H} + (\mathbf{A}-\mathbf{A}_0)\times\delta\mathbf{h}]\end{aligned}$$

where $\mathbf{A}$ and $\mathbf{A}_0$ are the vector potentials of the fields $\mathbf{B}$ and $\mathbf{B}_0 = \mu_2\mathbf{H}$ respectively. By virtue of (1.2) the integral of the second term vanishes. Denoting by

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$$\mathbf{m}=\frac{1}{4\pi}\int\limits_G(\mathbf{B}-\mu_2\mathbf{H})\,d\tau,\quad \mathbf{K}_1=\int\limits_G\mathbf{k}\,d\tau \tag{1.4}$$

the total magnetic and total internal mechanical moment of the system, we obtain

$$\delta\Phi=-\mathbf{m}\cdot\delta\mathbf{h}-\mathbf{K}_1\cdot\delta\mathbf{\Omega} \tag{1.5}$$

Formula (1.5) enables us to determine the dependence of Φ on $\mathbf{h}$ and $\mathbf{\Omega}$, provided that the quantities $\mathbf{m}$ and $\mathbf{K}_1$ have already been found from the solution of problem (1.2).

In the orthogonal $Ox^1x^2x^3$ coordinate system rigidly attached to the body, the components of the tensors are allocated the indices i, j, k. The symbols δ and $(\ )^{\cdot}$ denote the variation and the time derivative in this moving coordinate system.

Let

$$F_0(\mathbf{M})=\Theta^iM_i+{}^1/_2\Theta^{ij}M_iM_j+{}^1/_6\Theta^{ijk}M_iM_jM_k+\ldots \tag{1.6}$$

where Θ^i, Θ^{ij}, ... are given functions of the points of the body. Here and henceforth the repeated indices will denote summation except when they occur within the brackets.

Since the field $\mathbf{h}$ is uniform, the free energy of the body Φ depends, by virtue of (1.2)-(1.4), only on Ω_i and h^i. Then from (1.5) we obtain

$$\mathbf{m}=-\partial F/\partial\mathbf{h},\ \mathbf{K}_1=-\partial F/\partial\mathbf{\Omega} \tag{1.7}$$

2. Let $T=T(\mathbf{\Omega},\ \mathbf{u})$ be the kinetic energy of the fluid and the body, where $\mathbf{u}$ is the velocity of the point O. Then

$$T={}^1/_2\alpha_0{}^{ij}\Omega_i\Omega_j+\beta^{ij}u_j\Omega_i+{}^1/_2\gamma^{ij}u_iu_j$$

The motion of the body is determined using Hamilton's principle

$$\delta\int\limits_{t_1}^{t_2}L\,dt=0,\quad L=T-\Phi=L(\mathbf{\Omega},\mathbf{u},\mathbf{h}) \tag{2.1}$$

where the variation of the law of motion vanishes when $t=t_{1,2}$. Using the formulas for the variation

$$\delta\mathbf{\Omega}=(\delta\mathbf{\theta})^{\cdot}+\mathbf{\Omega}\times\delta\mathbf{\theta},\quad \delta\mathbf{u}=(\delta\mathbf{l})^{\cdot}+\mathbf{\Omega}\times\delta\mathbf{l}+\mathbf{u}\times\delta\mathbf{\theta},\quad \delta\mathbf{h}=\mathbf{h}\times\delta\mathbf{\theta}$$

where $\delta\mathbf{\theta}$, $\delta\mathbf{l}$ is the infinitesimal rotation and translational displacement of the body, we obtain

$$\begin{gathered}\delta_1\int\limits_{t_1}^{t_2}L\,dt=\int\limits_{t_1}^{t_2}\delta L\,dt+L\delta t\,|_{t_1}^{t_2}=\\ \int\limits_{t_1}^{t_2}\{(-\mathbf{K}^{\cdot}+\mathbf{K}\times\mathbf{\Omega}+\mathbf{p}\times\mathbf{u}+\mathbf{m}\times\mathbf{h})\cdot\delta\mathbf{\theta}+(-\mathbf{p}^{\cdot}+\mathbf{p}\times\mathbf{\Omega})\cdot\delta\mathbf{l}\}\,dt+\\ [\mathbf{K}\cdot\delta_1\mathbf{\theta}+\mathbf{p}\cdot\delta_1\mathbf{l}-\mathbf{H}\delta t]_{t_1}^{t_2},\quad \delta_1\mathbf{l}=\delta\mathbf{l}+\mathbf{u}\delta t,\quad \delta_1\mathbf{\theta}=\delta\mathbf{\theta}+\mathbf{\Omega}\delta t\end{gathered} \tag{2.2}$$

Here δ_1 is the total variation of the quantities when the law of motion and the time are both varied. The variables $\mathbf{p}=\partial L/\partial\mathbf{u}$, $\mathbf{K}=\partial L/\partial\mathbf{\Omega}$, $\mathbf{m}=\partial L/\partial\mathbf{h}$ are the momentum, the mechanical and magnetic moment of the system, and $\mathbf{H}=\mathbf{K}\cdot\mathbf{\Omega}+\mathbf{p}\cdot\mathbf{u}-L$ is the energy of the system.

Let us consider $\mathbf{H}$ as a function of the variables $\mathbf{K}$, $\mathbf{p}$, $\mathbf{h}$. Then $\mathbf{\Omega}=\partial H/\partial\mathbf{K}$, $\mathbf{u}=\partial H/\partial\mathbf{p}$, $\mathbf{m}=-\partial H/\partial\mathbf{h}$ and by virtue of (2.1), (2.2) we have

$$\mathbf{K}^{\cdot}=\mathbf{K}\times\partial H/\partial\mathbf{K}+\mathbf{p}\times\partial H/\partial\mathbf{p}+\mathbf{h}\times\partial H/\partial\mathbf{h},\quad \mathbf{p}^{\cdot}=\mathbf{p}\times\partial H/\partial\mathbf{K} \tag{2.3}$$

The equation

$$\mathbf{h}^{\cdot}=\mathbf{h}\times\partial H/\partial\mathbf{K} \tag{2.4}$$

describing the change in the vector $\mathbf{h}$ in the moving coordinate system, enables us to obtain a closed system of equations in $\mathbf{K}$, $\mathbf{p}$, $\mathbf{h}$.

3. We shall adopt certain simplifying assumptions when calculating Φ. We shall assume that the function $F_0(M)$ is given by the first two terms of expansion (1.6). Then the solution of problem (1.2) will depend linearly on $\mathbf{h}$ and $\mathbf{\Omega}$, and by virtue of (1.2) and (1.4) the quantities $\mathbf{m}$ and $\mathbf{K}_1$ will be written in the form

$$m_j=M_j-d_{ij}{}^{(1)}h^i-\eta_j{}^i\Omega_i,\quad K_1{}^i=\varkappa^i-\eta^i{}_jh^j+\alpha_1{}^{ij}\Omega_j \tag{3.1}$$

where M_j, $\varkappa^i$, $d_{ij}{}^{(1)}$, $\eta^i{}_j$, $\alpha_1{}^{ij}$ are constants determined by the geometry of the body and the quantities μ_2, Θ^i, Θ^{ij}, g. Substituting (3.1) into (1.5) we obtain, apart from the additive constant, the expression for the free energy

$$\Phi=-M_jh^j-\varkappa^i\Omega_i+{}^1/_2d_{ij}^{(1)}h^ih^j+\eta_j{}^i\Omega_ih^j-{}^1/_2\alpha_1{}^{ij}\Omega_i\Omega_j \tag{3.2}$$

In the general case the analytic determination of the constants occurring in expression (3.2) is fairly complicated. We shall examine some examples in which the constants can be written in explicit form.

Example 1. (cf. /5/). Let ∂G be an ellipsoid, $\theta^{ij} = \delta^{ij}\mu_1\chi_1^{-1}$, and the quantities g, θ^i, μ be constant in G. Then the exact solution of problem (1.2) is known /2/. Here the field $\mathbf{H}$ is constant within the body and has the following form in the principal axes of the ellipsoid:

$$H^i = (h^{(i)} + 4\pi N_{(i)}\Gamma^{(i)}\chi_1\mu_2^{-1})\, Z^{(i)}, \quad \Gamma^i = \theta^i + \Omega^i g^{-1}, \quad Z^i = (1 + 4\pi\chi N_i)^{-1}$$

Here χ_1 and χ_2 is the magnetic susceptibility of the fluid and the body respectively, $\chi = (\mu_1 - \mu_2)/4\pi\mu_2$, and N_i are the demagnetizing factors of the ellipsoid /2/. From (1.4) we obtain (V is the volume of the ellipsoid)

$$m^i = \mu_2 V\,(\chi h^{(i)} - \Gamma^{(i)}\chi_1\mu_2^{-1})\, Z^{(i)}, \quad K_1^{\;i} = -V\chi_1\,(h^{(i)} - \Gamma^{(i)}\,(1 - 4\pi N_{(i)}\chi_2\mu_2^{-1}))\, Z^{(i)}_{\cdot} g^{-1}$$

Therefore we have

$$\begin{aligned}
&d_{ij}^{(1)} = d_{(i)}^{(1)}\delta_{(i)j}, \quad \eta_j^{\;i} = \eta_{(j)}\delta_{(j)}^i, \quad \alpha_1^{\;ij} = \alpha_1^{(i)}\delta^{(i)j}\\
&\mathrm{M}_i = -\chi_1\theta_{(i)} V Z^{(i)}, \quad d_i^{(1)} = -\mu_2\chi V Z^i, \quad \eta_i = \chi_1 V Z^i g^{-1}\\
&\alpha_1^{\;i} = \chi_1 V g^{-2}\,(1 - 4\pi\chi_2\mu_2^{-1} N_{(i)})\, Z^{(1)}\\
&\varkappa^i = \chi_1\theta^{(i)} V g^{-1}\,(1 - 4\pi\chi_2\mu_2^{-1} N_{(i)})\, Z^{(i)}
\end{aligned}$$

Example 2. Let the form of the body be arbitrary, $\mu_1 = \mu_2$, and let g, θ^i be constant in G. Then the solution of problem (1.2) will have the form

$$\mathrm{H}^i = h^i - \chi_2\mu_2^{-1}\,(\Omega^j g^{-1} + \theta^j)\,\nabla^i\nabla_j P, \quad P(\mathbf{x}) = \int_G |\mathbf{x} - \mathbf{y}|^{-1}\, d\tau_y$$

By virtue of (1.4) and (3.1) (V is the volume of the body), we have

$$\begin{aligned}
&\mathrm{M}_i = -\chi_2\theta_i V, \quad \eta_j^{\;i} = \eta\delta_j^{\;i}, \quad \eta = \chi_2 V g^{-1}, \quad d_{ij}^{(1)} = 0\\
&\varkappa^i = \chi_2 g^{-1} T^{ij}\theta_j, \quad \alpha_1^{\;ij} = \chi_2 g^{-2} T^{ij}, \quad T^{ij} = \int_G (\delta^{ij} + \chi_2\mu_2^{-1}\nabla^i\nabla^j P)\, d\tau
\end{aligned}$$

We note that taking into account in expansion (1.6) the terms of second and higher degrees, leads to the appearance of terms of degree higher than the second in the expansion of Φ in terms of $\mathbf{h}$ and $\mathbf{\Omega}$.

4. Let us obtain an explicit expression for the function $H = H\,(\mathbf{K}, \mathbf{p}, \mathbf{h})$, using the assumptions of Sect.3. According to (2.1) and (3.2),

$$\begin{aligned}
L = T - \Phi = {}^1/_2\alpha^{ij}\Omega_i\Omega_j + \beta_{ij}\Omega_i u_j + {}^1/_2\gamma^{ij}u_i u_j -{}\\
\eta_j^{\;i}\Omega_i h^j - {}^1/_2 d_{ij}^{(1)} h^i h^j + \mathrm{M}_i h^i + \varkappa^i\Omega_i, \quad \alpha^{ij} = \alpha_0^{\;ij} + \alpha_1^{\;ij}
\end{aligned} \tag{4.1}$$

The total mechanical and magnetic moment, the momentum and the energy, have the form

$$\begin{aligned}
&K^i = \partial L/\partial\Omega_i = \alpha^{ij}\Omega_j + \beta^{ij}u_j - \eta_j^{\;i}h^j + \varkappa^i\\
&m_j = \partial L/\partial h^j = -d_{ij}^{(1)}h^i - \eta_j^{\;i}\Omega_i + \mathrm{M}_j\\
&p_i = \partial L/\partial u_i = \beta^{ij}\Omega^j + \gamma^{ij}u_j\\
&H = {}^1/_2 a_{ij}K^iK^j + b_{ij}K^ip_j + {}^1/_2 c_{ij}p^ip^j + {}^1/_2 d_{ij}h^ih^j +{}\\
&\quad e_{ij}K^ih^j + f_{ij}p^ih^j\\
&- a_{ij}\varkappa^iK^j - b_{ij}\varkappa^ip^j - (\mathrm{M}_i + e_{ij}K^j)\,h^i + {}^1/_2 c_{ij}\varkappa^i\varkappa^j\\
&\begin{Vmatrix} \|\alpha^{ij}\| & \|\beta^{ij}\| \\ \|\beta^{ij}\| & \|\gamma^{ij}\| \end{Vmatrix}^{-1} = \begin{Vmatrix} \|a_{ij}\| & \|b_{ij}\| \\ \|b_{ij}\| & \|c_{ij}\| \end{Vmatrix}\\
&d_{ij} = d_{ij}^{(1)} + a_{km}\eta_i^{\;k}\eta_j^{\;m}, \quad e_{ij} = a_{ik}\eta_j^{\;k}, \quad f_{ij} = b_{ik}\eta_j^{\;k}
\end{aligned} \tag{4.2}$$

By virtue of (4.1) the tensor $\alpha_1^{\;ij}$ which plays the part of the associated inertia tensor governed by the magnetic properties of the material, exerts a significant influence on the motion of the rigid body G.

Thus, for example, when $\chi_1 > 0$, the matrix $\|\alpha_1^{\;ij}\|$ is positive definite in the case of an ellipsoidal body (example 1), therefore additional work is needed to start the rotation of the body, equal in magnitude when $\varkappa^i = 0$ to the energy of the system Φ in the magnetic field formed when the body rotates. It should be noted that the tensor $\alpha_1^{\;ij}$ is not, generally speaking, positive definite.

In the case of ferromagnetic materials $M_i = 0$ if and only if $\varkappa^i = 0$. This can be explained by the fact that when the electron spins are aligned in any one direction, then intrinsic non-zero magnetic and mechanical moments appear even when the body is at rest in a zero external magnetic field. In general $M_i = 0$ does not result in $\varkappa_i = 0$ due, for example, to the presence of constant ring currents within the body.

The quantities $e_{ij}, f_{ij}, \varkappa^i$ vanish as $g \to \infty$, i.e. the presence of the corresponding terms in (4.2) is essentially connected with the appearance of gyromagnetic effects.

Let $M_i = 0$, $\varkappa^i = 0$. We denote by $K_C{}^i = \alpha^{ij}\Omega_j + \beta^{ij}u_j$, $m_{Cj} = -\Omega_i\eta_j{}^i$ the mechanical and magnetic moment of the system due to the motion of the body, and $K_1{}^i = -\eta_j{}^ih^j$, $m_{1j} = -d_{ij}{}^{(1)}h^i$ are the moments connected with the presence of an external field. We shall write Eqs.(2.3) and (2.4) in the form

$$\mathbf{K}_C^{\cdot} = \mathbf{K}_C \times \boldsymbol{\Omega} + \mathbf{p}\times\mathbf{u} + \mathbf{m}_1\times\mathbf{h} + \mathbf{M}_g, \quad \mathbf{p}^{\cdot} = \mathbf{p}\times\boldsymbol{\Omega}, \quad \mathbf{h}^{\cdot} = \mathbf{h}\times\boldsymbol{\Omega}$$

where $\mathbf{M}_g = \mathbf{K}_1 \times \boldsymbol{\Omega} + \mathbf{m}_C \times \mathbf{h} - \mathbf{K}_1^{\cdot}$ is the moment of forces acting on the mechanical system "body + fluid" from the direction of the field and depending on gyromagnetic effects. Let us transform $\mathbf{M}_g$ to the form

$$\mathbf{M}_g = -(\Lambda\mathbf{h})\times\boldsymbol{\Omega} - (\boldsymbol{\Omega}\Lambda)\times\mathbf{h} + \Lambda(\mathbf{h}\times\boldsymbol{\Omega}), \quad \Lambda = (\eta_j{}^i) \tag{4.3}$$

Since $\langle \mathbf{M}_g, \boldsymbol{\Omega}\rangle = 0$, it follows that the forces related to the gyromagnetic effects are gyroscopic. If $\eta_j{}^i = \eta_{(j)}\delta_{(j)}{}^i$, then $M_g{}^i = e^{(i)jk}h_j\Omega_k(\eta_{(i)} + \eta_k - \eta_j)$. In particular, when $\eta_i = \eta$ we obtain

$$\mathbf{M}_g = -\eta\boldsymbol{\Omega}\times\mathbf{h} \tag{4.4}$$

The rotation of a rigid body with a fixed point acted upon by a force moment of the form (4.4) was studied in /6/.

5. We introduce, in the space of infinitely differentiable functions defined on $\mathbf{R}^9(\mathbf{K}, \mathbf{p}, \mathbf{h})$, the Poisson bracket, assuming that

$$\begin{aligned} &\{K^i, K^j\} = -e^{ijk}K^k, \ \{K^i, p^j\} = -e^{ijk}p^k, \ \{K^i, h^j\} = -e^{ijk}h^k \\ &\{h^i, h^j\} = \{p^i, h^j\} = \{p^i, p^j\} = 0 \end{aligned} \tag{5.1}$$

Then we can rewrite system (2.3), (2.4) in the form /7/

$$\mathbf{K}^{\cdot} = \{\mathbf{K}, H\}, \quad \mathbf{p}^{\cdot} = \{\mathbf{p}, H\}, \quad \mathbf{h}^{\cdot} = \{\mathbf{h}, H\} \tag{5.2}$$

We note that a bracket of the type (5.1) was first used in /8/ while studying the Hamiltonian structure of the Kirchhoff equations.

The functions $I_1 = \mathbf{p}^2$, $I_2 = \mathbf{p}\cdot\mathbf{h}$, $I_3 = \mathbf{h}^2$ commute with any smooth function on $\mathbf{R}^9(\mathbf{K}, \mathbf{p}, \mathbf{h})$, i.e. the bracket (5.1) is degenerate. From (5.2) it follows that I_1, I_2, I_3 are first integrals. Restriction of the bracket (5.1) to their non-singular compatible level

$$I_{123}(l_1, l_2, l_3) = \{I_1 = l_1^2 > 0, \ I_2 = l_2, \ I_3 = l_3^2 > 0\}$$

is non-degenerate and can be reduced for certain special variables to canonical form.

The dimensionality of I_{123} is equal to six, therefore the Liouville integrability of system (5.2) requires two additional mutually commutative first integrals.

Let the level of the integrals I_{123} be singular. Then $l_2 = l_1l_3$, and a vector $\boldsymbol{\gamma}$: $\mathbf{p} = l_1\boldsymbol{\gamma}$, $\mathbf{h} = l_3\boldsymbol{\gamma}$ exists. At this level the Hamiltonian function (4.2) and the Poisson bracket (5.1) take the form

$$H = {}^1/_2A_{ij}K^iK^j + B_{ij}K^i\gamma^j + {}^1/_2C_{ij}\gamma^i\gamma^j + D_iK^i + E_i\gamma^i \tag{5.3}$$

$$\{K^i, K^j\} = -e^{ijk}K^k, \quad \{K^i, \gamma^j\} = -e^{ijk}\gamma^k, \quad \{\gamma^i, \gamma^j\} = 0 \tag{5.4}$$

$$\begin{aligned} &A_{ij} = a_{ij}, \quad B_{ij} = l_1b_{ij} + l_3e_{ij}, \quad C_{ij} = l_1^2c_{ij} + l_3^2d_{ij} + l_1l_3f_{ij} \\ &D_i = -a_{ij}\varkappa^j, \quad E_i = -l_1b_{ij}\varkappa^j - l_3(\mathrm{M}_i + e_{ij}\varkappa^j) = -B_{ij}\varkappa^j - l_3\mathrm{M}_i \end{aligned}$$

The corresponding Hamilton equations

$$\mathbf{K}^{\cdot} = \{\mathbf{K}, H\}, \quad \boldsymbol{\gamma}^{\cdot} = \{\boldsymbol{\gamma}, H\} \tag{5.5}$$

describe, in the limit when $l_1 = 0$, the motion of the body about a fixed point.

System (5.5) has three first integrals, namely the energy integral $J_0 = H$, the area integral $J_1 = \mathbf{K}\cdot\boldsymbol{\gamma}$, and the integral $J_2 = \boldsymbol{\gamma}^2$. Its full integrability requires another integral.

There are certain cases when such an integral exists. These are the Euler, Lagrange, Kowalewska and Goryachev-Chaplygin cases from the problem of motion of a heavy rigid body about a fixed point, the Zhukovskii, Sretenskii and the dynamic symmetry cases from the problem of the motion of a gyrostat, and the dynamic symmetry, Klebsch, Lyapunov, Steklov and Chaplygin cases from the problem of the motion of a rigid body in an ideal fluid.

Let $a_{ij} = a_{(i)}\delta_{(i)j}$, a_1, a_2, a_3 be different and positive. If $\mathbf{B} = \mathbf{C} = 0$, $\mathbf{D} = 0$, then according to /9, 10/ there is no additional integral in the class of analytic functions when $\mathbf{E} \neq 0$. When $\mathbf{B} = 0$, $\mathbf{D} = 0$, $\mathbf{E} = 0$, then an additional integral exists only in the Klebsch case /11/. If $\mathbf{B} \neq 0$, $\mathbf{D} = 0$, $\mathbf{E} = 0$, then the necessary conditions for the existence of an additional analytic integral have the form /11/

$$B = \operatorname{diag}(b_1, b_2, b_3),\quad a_1^{-1}(b_2 - b_3) + a_2^{-1}(b_3 - b_1) + a_3^{-1}(b_1 - b_2) = 0 \tag{5.6}$$

When $M = \varkappa = 0$, then the Hamiltonian function for the case discussed in Sect.2 has the form

$$H = {}^1/_2 a_i K_i^2 + \eta a_i K_i \gamma_i + {}^1/_2 \eta^2 a_i \gamma_i^2 \tag{5.7}$$

Then, provided that a_1, a_2, a_3 are different, the condition for the existence of an additional integral (5.6) can be reduced to the form $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = 0$. Therefore, the equations of motion can be integrated if and only if two of the a_i are the same, e.g. $a_1 = a_2$. Here the additional integral has the form $J_3 = K_3$.

6. Let us consider the stationary motions of the body in the case discussed in example 2, assuming that $M = \varkappa = 0,\ l_1 = 0$. If $a_1 \neq a_2 \neq a_3$, then the stationary motions will be determined by the relation

$$\delta U_1 = 0,\quad U_1 = H + \lambda J_1 + {}^1/_2 \mu J_2 \quad (\eta = 1,\ \mathbf{h} \equiv \gamma)$$

The stationary motions represent uniform rotations about one of the axes of inertia of the body, and the conditions of stability of rotation about the i-th axis have the form $a_j > a_i,\ j \neq i$.

By virtue of the hydrodynamic analogy, we have the correspondence between the permanent rotations of a gyromagnetic rigid body and the constant screw motions of a body in a fluid. The stability of such motions was studied by Lyapunov /13/.

When the body has dynamic symmetry $(a_1 = a_2 = a)$, we have in the class of stationary motions in addition to the permanent motions, also regular precessions. The stationary motions are found from the condition

$$\delta U_2 = 0,\quad U_2 = H + \lambda J_1 + {}^1/_2 \mu J_2 - \omega J_3$$

The above condition has the form

$$aK_\alpha + (a + \lambda) h_\alpha = 0,\quad (a + \lambda) K_\alpha + (a + \mu) h_\alpha = 0,\quad \alpha = 1, 2 \tag{6.1}$$

$$a_3 K_3 + (a_3 + \lambda) h_3 = \omega,\quad (a_3 + \lambda) K_3 + (a_3 + \mu) h_3 = 0 \tag{6.2}$$

System (6.1) admits of a non-trivial solution $(K_\alpha^\circ, h_\alpha^\circ)$, provided that condition $a(a + \mu) - (a + \lambda)^2 = 0$ holds. Obtaining from it the expression for μ and substituting it into the second equation of (6.2), we obtain the equation

$$(a_3 + \lambda) K_3 + (a_3 - a + a^{-1}(a + \lambda)^2) h_3 = 0 \tag{6.3}$$

connecting the value of λ with the values of $K_3 = K_0, h_3 = h_0$ preserved on the regular precessions. When K_0, h_0 are given, a regular precession exists provided that Eq.(6.3) can be solved for λ, i.e., when the following condition holds:

$$aK_0^2 - 4h_0(a_3 - a)(K_0 + h_0) \geqslant 0 \tag{6.4}$$

The regular precession is described by the equations

$$K_1^\circ = \Lambda h_* \sin \omega t,\quad h_1^\circ = h_* \sin \omega t,\quad K_2^\circ = \Lambda h_* \cos \omega t,\quad h_2^\circ = h_* \cos \omega t$$

$$K_3 = K_0,\quad h_3 = h_0,\quad h_* = \sqrt{h^2 - h_0^2},\quad \Lambda = -(a + \lambda)/a$$

where h is the strength of the external magnetic field.

It can be shown that when the condition

$$a(K_0 - 2\Lambda h_0)^2 + (a_3 - a + a\Lambda^2)(h^2 - h_0^2) > 0$$

holds, the corresponding precessional motion is stable in the variables $u_\alpha = K_\alpha - \Lambda h_\alpha,\ \alpha = 1, 2, h_3, K_3$.

In the limit, as $h_0 \to h$, the regular precession transforms into a permanent rotation of the body about the axis of dynamic symmetry. The condition of stability of this rotation with respect to the variables K_i, h_i has the form

$$aK_0^2 - 4h(a_3 - a)(K_0 + h) > 0 \tag{6.5}$$

We note that according to (6.5) the stability of rotation of a gyromagnetic rigid body about the axis of dynamic symmetry depends essentially not only on the magnitude of the angular velocity, but also on the direction of rotation.

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ON THE ORBITAL STABILITY OF A PERIODIC SOLUTION OF THE EQUATIONS OF MOTION OF A KOVALEVSKAYA GYROSCOPE*

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The sufficient conditions for the orbital stability of a periodic solution of the equations of motion of a Kovalevskaya gyroscope in the case of Bobylev-Steklov integrability are obtained.

It is difficult to expect Lyapunov stability for the unsteady motions of a heavy solid having a fixed point since a dependence of the vibrations frequency on the initial conditions is characteristic for the simplest of them, i.e. periodic motions /1/. Moreover, a rougher property of periodic solutions of the Euler-Poisson equations, orbital stability /2/, is not the subject of special investigations in the dynamics of a solid. The algorithm of the present investigation utilizes the treatment ascribed Zhukovskii /3/ of orbital stability as the Lyapunov stability of motion for a special selection of the variable playing the part of time (see /4/ also) and the Chetayev method /5/ of constructing Lyapunov functions from the first integrals of the equations of perturbed motion. This latter circumstance enables the Chetayev method to be put in one series with the methods used in /1, 4, 6-9/, etc.

1. Under the Kovalevskaya conditions the Euler-Poisson equations and the first integrals have the following form in dimensionless variables /10/

$$2p^\cdot = qr,\quad 2q^\cdot = -rp - \gamma'',\quad r^\cdot = \gamma' \tag{1.1}$$
$$\gamma^\cdot = \gamma' r - \gamma'' q,\quad \gamma'^\cdot = \gamma'' p - \gamma r,\quad \gamma''^\cdot = \gamma q - \gamma' p$$
$$2(p^2 + q^2) + r^2 - 2\gamma = 6l_1,\quad 2(p\gamma + q\gamma') + r\gamma'' = 2l \tag{1.2}$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1,\quad (p^2 - q^2 + \gamma)^2 + (2pq + \gamma')^2 = k^2$$

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